INTEGRAL METHOD OF CALCULATING A TURBULENT BOUNDARY LAYER IN A UNIFORM GAS STREAM

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UDC 532.526.4

A previously developed integral method for calculating a turbulent boundary layer with pressure gradient and blowing [1] has been generalized to the case of flow with heat transfer.

Calculation of flow in a turbulent boundary layer with longitudinal pressure drop, blowing, and heat transfer is a matter of considerable interest. In spite of the appearance in recent years of finite-difference methods for integrating the turbulent boundary-layer equations, integral methods are preferable in many cases.

Most of the well-known integral methods are based on assigning some a priori approximation for the velocity profile or the shear stress in the turbulent boundary layer. A somewhat different approach to assigning a family of velocity profiles was suggested in [1]. The velocity profile in the boundary layer is obtained by integrating the turbulent boundary-layer equations separately in the inner and outer regions, accounting for flow peculiarities in each zone. Here the only assumption is that universal relations hold for the velocity profile in each region, and the specific form of these relations is not specified.

In the present paper the integral method proposed in [1] is generalized to the case of flow with heat transfer.

The averaged equations of continuity, momentum transfer, and energy describing steady-state two-dimensional flow in a turbulent boundary layer in a compressible gas, on a permeable surface, and the boundary conditions have the form [2]

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)=0, \tag{1}$$

$$\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}=\rho_e u_e\frac{du_e}{dx}+\frac{\partial \tau}{\partial y},\quad \tau=(\mu+\varepsilon)\frac{\partial u}{\partial y}, \tag{2}$$

$$\rho u\frac{\partial h}{\partial x}+\rho v\frac{\partial h}{\partial y}=\rho_e u_e\frac{du_e}{dx}u+(\mu+\varepsilon)\left(\frac{\partial u}{\partial y}\right)^2+\frac{\partial}{\partial y}\left[\left(\frac{\mu}{\mathrm{Pr}}+\frac{\varepsilon}{\mathrm{Pr}_\tau}\right)\frac{\partial h}{\partial y}\right], \tag{3}$$

$$\frac{\rho}{\rho_e}=\frac{T_e}{T},\quad h=\int_0^T c_p dT,\quad \mathrm{Pr}=\frac{\mu c_p}{\lambda},\quad \mathrm{Pr}_\tau=\frac{\varepsilon c_p}{\lambda_\tau}, \tag{4}$$

$$u=0,\quad v=v_w,\quad h=h_w\quad \text{at}\quad y=0, \tag{5}$$

$$u\to u_e,\quad h\to h_e\quad \text{at}\quad y\to\infty. \tag{6}$$

Uniformity of the gas stream presupposes that there are no chemical reactions and that the gas blown into the boundary layer is identical in composition with the main flow gas.

Following the approach adopted in [1], we shall divide the entire boundary layer into two regions: an inner region in which the law of the wall holds

$$\varphi=\varphi(\eta),\quad \eta=\rho_w y v_*/\mu_w,\quad \varphi=u/v_*,\quad v_*=(\tau_w/\rho_w)^{1/2}, \tag{7}$$

and an outer region where the wake law holds.

Rybinskii Aviation-Technology Institute. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 34, No. 3, pp. 500-506, March, 1978. Original article submitted February 7, 1977.

For the turbulent viscosity in the inner region (ε_i) and in the wake law region (ε_0) we use modified Prandtl–Van Dreist and Clauser relations, respectively [3]:

$$\varepsilon_i = \mu\varkappa^2\eta^2[1-\exp(-\eta/A)]^2 d\varphi/d\eta,$$

$$A = 26\frac{\mu}{\mu_w}\left(\frac{\rho_w}{\rho}\right)^{1/2}\left[\frac{\mu}{\mu_e}\left(\frac{\rho_e}{\rho_w}\right)^2\frac{P_*}{B_*}\left(1-e^{\frac{\mu B_*}{\mu_w}\eta_L}\right)+e^{\frac{\mu B_*\eta_L}{\mu_w}}\right], \tag{8}$$

$$P_* = \frac{\mu_e u_e}{\rho_e v_*^3}\frac{du_e}{dx}, \quad B_* = \frac{v_w}{v_*},$$

$$\varepsilon_0 = k\rho_e u_e\int_0^\infty (1-U)\,dy, \quad U = u/u_e, \quad k = 0.0168. \tag{9}$$

The value of the variable η at the laminar sublayer boundary η_L is given by the transcendental relation

$$\left[1-\frac{\mu_w}{\mu_e}\left(\frac{\rho_e}{\rho_w}\right)^2\frac{P_*}{B_*}\right]\exp(B_*\eta_L)+\frac{\mu_w}{\mu_e}\left(\frac{\rho_e}{\rho_w}\right)^2\frac{P_*}{B_*} = \frac{\alpha^2}{\eta_L^2}. \tag{10}$$

To calculate the molecular gas viscosity μ we used the power law [2]

$$\frac{\mu}{\mu_e} = \left(\frac{T}{T_e}\right)^{0.76}. \tag{11}$$

In accordance with the turbulent boundary layer model adopted, we shall start from the flow peculiarities in each region, in constructing a family of velocity profiles. Here we use a technique discussed in detail in [1].

In the inner region the equation of motion (2) is integrated formally over the transverse coordinate, assuming that the law of the wall (7) holds and that the density and velocity profiles in the boundary layer are related by the modified Crocco relation [2]:

$$\frac{\rho}{\rho_w} = \left(1-\frac{\omega}{\zeta}\varphi-\frac{\beta}{\zeta^2}\varphi^2\right)^{-1}, \quad \omega = 1-\frac{T_r}{T_w}, \quad \beta = r\frac{\gamma-1}{2}M_e^2\frac{T_e}{T_w}, \tag{12}$$

where ζ is the friction parameter; and T_r is the recovery temperature.

The resulting expression for the shear stresses has the form

$$\frac{\tau}{\tau_w} = 1+BU+\frac{\mu_w\zeta^3}{\rho_w v_*}\left[\frac{1}{v_*}\frac{dv_*}{dx}K_{12}-\frac{\eta\rho_e}{\zeta\rho_w u_e}\frac{du_e}{dx}+\right.$$
$$\left.+\left(\frac{d\omega}{dx}-\frac{\omega}{\zeta}\frac{d\zeta}{dx}\right)(K_{23}-UK_{22})+\left(\frac{d\beta}{dx}-\frac{2\beta}{\zeta}\frac{d\zeta}{dx}\right)(K_{24}-UK_{23})\right],$$
$$K_{mn} = \int_0^U\left(\frac{\rho}{\rho_w}\right)^m U^n\frac{d\eta}{d\varphi}\,dU, \quad B = \zeta B_*. \tag{13}$$

Here by ρ/ρ_w we mean not the approximate relation (12), but the more accurate dependence of density on velocity which will be obtained below. Thus, the Crocco integral is used in the present method only to derive an explicit form of Eq. (13).

By substituting the expression for the turbulent viscosity (8) into Eq. (2) for the shear stresses, following some transformations, we obtain a single relation between the velocity profiles and the shear stresses in the inner boundary-layer region

$$\eta = \int_0^\varphi\frac{2\frac{\rho}{\rho_w}\varkappa^2\eta^2(1-e^{-\eta/A})^2}{\left[\left(\frac{\mu}{\mu_w}\right)^2+4\frac{\rho}{\rho_w}\varkappa^2\eta^2(1-e^{-\eta/A})^2\frac{\tau}{\tau_w}\right]^{1/2}-\frac{\mu}{\mu_w}}\,d\varphi. \tag{14}$$

For a fixed value of the friction factor and with the known relation between the density and velocity, Equations (13) and (14) describe the velocity profile in the inner region of the turbulent boundary layer of a compressible gas.

Following the approach used in [1], the boundary-layer equation of motion in the inner region, in Von Mises type variables

$$z=\psi/\xi^{1/2},\quad \xi=\int_0^x \varepsilon_0\rho_e u_e dx \tag{15}$$

is integrated in the local similarity approximation with boundary conditions corresponding to mixing of the two flows:

$$\frac{U''}{U'}+\frac{U'}{U}+\frac{\rho_0'}{\rho_0}+\frac{z}{2\rho_0 U}+\left(\frac{1}{\rho_0 U^2}-1\right)\frac{\beta_0}{\rho_0 U'}=0,\quad \beta_0=\frac{\xi}{u_e}\frac{du_e}{d\xi}, \tag{16}$$

$$U\to 1\,(z\to\infty),\quad U\to U_s\,(z\to-\infty). \tag{17}$$

Here U_s is a free parameter, chosen from the matching condition; and primes denote derivation with respect to z.

From the integration we obtain relations describing a family of velocity profiles in the wake law region:

$$U=1-(1-U_s)\frac{I(+\infty)-I(z)}{I(+\infty)-I(-\infty)}, \tag{18}$$

$$I(z)=\int_0^z \frac{1}{\rho_0 U}\exp\left\{-\int_0^z\left[\frac{z}{2\rho_0 U}+\left(\frac{1}{\rho_0 U^2}-1\right)\frac{\beta_0}{\rho_0 U'}\right]dz\right\}dz. \tag{19}$$

The velocity profile in the inner region is compared with one of the curves of the family (18). In accordance with the technique suggested in [1], the matching of the two profiles is performed in terms of the velocity and its first derivative with respect to the transverse coordinates. The position of the point of comparison is determined from the condition that the turbulent viscosity be continuous.

For convenience of later analysis we convert to the Crocco variables x and U in the energy equation (3), and restrict ourselves in this examination to the local similarity hypothesis. The possibility of a similarity approach has been discussed frequently in the literature [2, 4]. The estimates made in [4], together with numerous practical calculations in [2], confirm that local similarity is a permissible approximation in establishing the relation between the enthalpy and velocity profiles in a turbulent boundary layer. The dimensionless energy equation resulting from the similarity approximations in Crocco variables has the form

$$\left(\frac{H'}{\mathrm{Pr_{eff}}}\right)'+(1-\mathrm{Pr_{eff}})\frac{(\tau/\tau_w)'}{\tau/\tau_w}\frac{H'}{\mathrm{Pr_{eff}}}=-\frac{u_e^2}{h_e}. \tag{20}$$

Here the primes denote derivation with respect to U. By introducing the effective Prandtl number $\mathrm{Pr_{eff}}=(\mu+\varepsilon)/(\mu\mathrm{Pr}^{-1}+\varepsilon\mathrm{Pr}_T^{-1})$ we can consider the energy equation in the form of Eq. (20) directly for the entire boundary-layer region, without dividing into a laminar sublayer and a turbulent core. Here the nonlinear nature of interaction between the molecular and molar transport processes is described by the Van Dreist damping factor, introduced into the expression for the turbulent viscosity.

By integrating Eq. (20) with boundary conditions (5) and (6), we obtain relations linking the enthalpy and the velocity in the turbulent boundary layer:

$$\bar H(U)=H_w+(1-\bar H_w)\frac{S(1)}{S(U)}+\frac{u_e^2}{h_e}\left[\frac{S(1)}{S(U)}R(1)-R(U)\right], \tag{21}$$

$$L(U)=\exp\left[-\int_1^{\tau/\tau_w}\frac{1-\mathrm{Pr_{eff}}}{\tau/\tau_w}d\left(\frac{\tau}{\tau_w}\right)\right], \tag{22}$$

$$S(U)=\int_0^U \mathrm{Pr_{eff}}L(U)\,dU,\quad R(U)=\int_0^U\left[\mathrm{Pr_{eff}}L(U)\int_0^U\frac{dU}{L(U)}\right]dU. \tag{23}$$

From the relations obtained for the enthalpy profile we derive a formula to calculate the recovery factor [2]

$$r = 2R(1). \tag{24}$$

The only undetermined parameter in the expressions for the velocity and enthalpy profiles is the friction factor c_f. To obtain it we consider the integral momentum relation in the form

$$\frac{d\theta}{dx} = \rho_e u_e \frac{c_f}{2}(1+B)e^F, \quad \theta = \rho_e u_e \delta^{**} e^F,$$
$$F = \int\limits_{x_t}^{x} \frac{du_e}{dx}\frac{1+H^*}{u_e}dx, \quad H^* = \frac{\delta^*}{\delta^{**}}, \quad \delta^* = \int\limits_0^\infty (1-\rho_0 U)\,dy, \tag{25}$$
$$\delta^{**} = \int\limits_0^\infty \rho_0 U(1-U)\,dy.$$

Integrating Eq. (25) with respect to x, substituting the transformed expression for the momentum thickness into it, and solving the relation obtained for c_f, we have

$$c_f = 2\mu_w e^F \int\limits_0^{U_m} \rho_0 U(1-U)\frac{d\eta}{d\varphi}dU\left[\frac{1}{2}\int\limits_{x_t}^{x} c_f \rho_e u_e (1+B)e^F dx - \xi^{1/2}e^F \int\limits_{z_m}^{\infty}(1-U)\,dz + \rho_{et}u_{et}\delta_t^{**}\right]^{-1}. \tag{26}$$

Here the subscripts m and t denote values of the quantities at the points of comparison of the velocity profile and at the initial section, respectively.

With Eq. (26), the expressions obtained earlier for the velocity profiles, Eqs. (14) and (18), and the enthalpy profile, Eq. (21), we can set up a process of successive approximations to evaluate the friction coefficient c_f.

With this method turbulent boundary layer flows were calculated for various conditions at the surface and at the outer edge. The gas specific heat c_p was considered to be constant in all the calculations. We assumed the following values for the molecular and turbulent Prandtl numbers: Pr = 0.72; $\mathrm{Pr_T}$ = 0.9-1.0 (air). The parameters at the outer edge of the boundary layer were calculated from the isentropic relations.

Figure 1 shows the calculated friction coefficient c_f in the turbulent boundary layer on a thermally insulated flat plate for three values of Mach number M_e, in comparison with the test data of [5]. Good agreement between the theoretical and experimental relations can be seen.

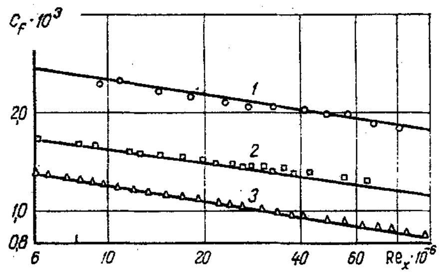

Fig. 1

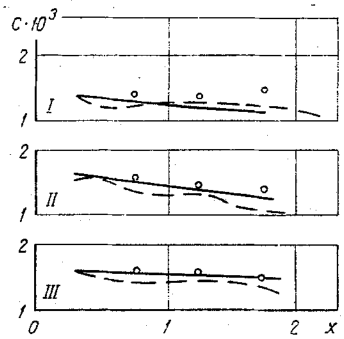

Fig. 2

Fig. 1. Friction coefficient on a thermally insulated flat plate, for three values of Mach number: 1) M = 2.0; 2) 2.95; 3) 4.2.

Fig. 2. Friction coefficient for flows with a strong (I), moderate (II), and weak (III) positive pressure gradient. The solid curves are for the integral method, and the broken curves are for the numerical method of [6]; x is in cm.

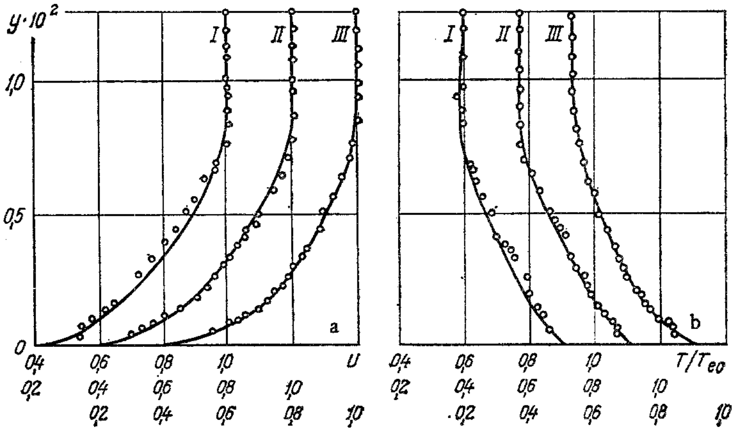

Fig. 3. Profiles of velocity (a) and temperature (b) through a turbulent boundary layer with a strong (I), moderate (II), and weak (III) positive pressure gradient; y is in m.

Figure 2 compares the theoretical and experimental values of friction coefficient for three values of positive pressure gradient differing in magnitude and nature. The theoretical curves are in quite satisfactory agreement with the experimental data of [6]. The maximum deviation is observed for the largest pressure gradient and is roughly 20% at the section x = 17.5 cm. It is interesting to note that the results obtained here using the integral method agreed with the experimental results fully as well as the curves taken from [6] which correspond to numerical solution of the turbulent boundary layer equations.

The velocity and temperature profiles for the three flows considered are shown in Fig. 3a, b. As one would expect, the largest deviation from the test data is observed in case I (the greatest pressure gradient).

In general it should be noted that the integral method described leads to quite reliable results in calculating turbulent boundary-layer flows with heat transfer, giving an accuracy, which, as a rule, is no worse than that from numerical solution of the turbulent boundary-layer equations.

NOTATION

x, y	are the coordinates directed along the surface and along the normal;
u, v	are the longitudinal and transverse velocity component;
ρ, c_p, T, h	are the density, specific heat, temperature, and enthalpy of the gas;
μ, λ, ε, λ_T	are the molecular and turbulent viscosity and thermal conductivity;
Pr, Pr_T, Pr_{eff}	are the molecular, tubulent, and effective Prandtl numbers;
φ, η	are the variables of the law of the wall;
v_*	is the dynamic velocity;
τ	is the friction shear stress;
P_*, B_*	are the pressure gradient and blowing parameters;
k, α, $\varkappa$	are the turbulence constants;
U	is the dimensionless velocity;
$H = h/h_e$	is the dimensionless enthalpy;
r	is the recovery factor;
$T_r = Te[1 + r(\gamma-1)/2) M_e^2]$	is the recovery temperature;
M	is the Mach number;
ω, β, ζ	are the heat transfer, compressibility, and friction parameters;
z, ξ	are the Von Mises type variables;
ψ	is the stream function;
$\rho_0 = \rho/\rho_e$	is the dimensionless density;
β_0	is the pressure gradient parameter;
$H^* = \delta^*/\delta^{**}$	is the shape parameter;
δ^*, δ^{**}	are the displacement and momentum thicknesses;
U_s	is the parameter for a family of jet velocity profiles;
c_f	is the friction coefficient.

Indices

e is the outer edge of the boundary layer;
w is the body surface;
t is the transition point;
L is the laminar sublayer boundary;
i is the inner sublayer;
0 is the outer sublayer.

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ENTRAINMENT OF A VISCOPLASTIC FLUID BY A MOVING SURFACE

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UDC 532.135

The thickness of the film remaining on the surface of a vertical plate during its extraction from a viscoplastic liquid is determined theoretically.

One of the most widespread methods of superposing a layer of lubricating fluid on a solid in different technological processes is to extract the solid from the fluid at a constant velocity v_0. Processes to obtain photographic materials, magnetic recorder tapes, cable insulation, etc. are examples.

Let an infinite plate be extracted vertically upward at a constant velocity v_0 from a sufficiently large vessel with a fluid. Far from the plate the fluid is at rest and its surface is horizontal. Let us take this horizontal surface as the origin x = 0 and let us direct the y axis perpendicularly to the plate and the x axis upward in the direction of plate motion.

The thickness of the film remaining on the plate surface as it is extracted from the fluid is determined by the interaction between the internal friction forces, the mass forces, and the surface tension force. The degree of influence of each of these forces on the quantity of fluid being entrapped is determined by the physical properties of the fluid, the state of the surface, the velocity of plate extraction, and a number of other factors.

According to the Landa–Levich–Deryagin theory [1, 2], the whole film can be separated into two domains: 1) a zone located sufficiently high above the meniscus and entrained directly by the body (the free boundary of the fluid is almost parallel to the plane of the plate in this domain); 2) the zone of the meniscus, which is deformed somewhat because of the plate motion (the shape of the surface is taken approximately coincident with the static meniscus). The solutions obtained for each domain must then be joined, where the junction condition is continuity of the surface curvature in the domain of small curvatures.

Following this path, we determine the thickness of the film remaining on the plate surface during its extraction from a viscoplastic fluid.

Thus, we have the static meniscus equation for zone 2:

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Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 34, No. 3, pp. 507-513, March, 1978. Original article submitted March 14, 1977.

0022-0841/78/3403-0344 $07.50